

**FINAL JEE-MAIN EXAMINATION – FEBRUARY, 2021**  
(Held On Friday 26<sup>th</sup> February, 2021) TIME : 3 : 00 PM to 6 : 00 PM

**MATHEMATICS**

**SECTION-A**

1. If vectors  $\vec{a}_1 = x\hat{i} - \hat{j} + \hat{k}$  and  $\vec{a}_2 = \hat{i} + y\hat{j} + z\hat{k}$  are collinear, then a possible unit vector parallel to the vector  $x\hat{i} + y\hat{j} + z\hat{k}$  is

(1)  $\frac{1}{\sqrt{2}}(-\hat{j} + \hat{k})$       (2)  $\frac{1}{\sqrt{2}}(\hat{i} - \hat{j})$   
(3)  $\frac{1}{\sqrt{3}}(\hat{i} + \hat{j} - \hat{k})$       (4)  $\frac{1}{\sqrt{3}}(\hat{i} - \hat{j} + \hat{k})$

**Official Ans. by NTA (4)**

**Sol.**  $\vec{a}_1$  and  $\vec{a}_2$  are collinear

so  $\frac{x}{1} = \frac{-1}{y} = \frac{1}{z}$

unit vector in direction of

$x\hat{i} + y\hat{j} + z\hat{k} = \pm \frac{1}{\sqrt{3}}(\hat{i} - \hat{j} + \hat{k})$

2. Let  $A = \{1, 2, 3, \dots, 10\}$  and  $f : A \rightarrow A$  be defined as  $f(k) = \begin{cases} k+1 & \text{if } k \text{ is odd} \\ k & \text{if } k \text{ is even} \end{cases}$

Then the number of possible functions  $g : A \rightarrow A$  such that  $g \circ f = f$  is

(1)  $10^5$       (2)  ${}^{10}C_5$       (3)  $5^5$       (4)  $5!$

**Official Ans. by NTA (1)**

**Sol.**  $f(x) = \begin{cases} x+1, & \text{if } x \text{ is odd} \\ x, & \text{if } x \text{ is even} \end{cases}$

$\therefore g : A \rightarrow A$  such that  $g(f(x)) = f(x)$

$\Rightarrow$  If  $x$  is even then  $g(x) = x$  ... (1)

If  $x$  is odd then  $g(x+1) = x+1$  ... (2)

from (1) and (2) we can say that

$g(x) = x$  if  $x$  is even

$\Rightarrow$  If  $x$  is odd then  $g(x)$  can take any value in set  $A$

so number of  $g(x) = 10^5 \times 1$

**TEST PAPER WITH SOLUTION**

3. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined as

$$f(x) = \begin{cases} 2 \sin\left(-\frac{\pi x}{2}\right), & \text{if } x < -1 \\ |ax^2 + x + b|, & \text{if } -1 \leq x \leq 1 \\ \sin(\pi x), & \text{if } x > 1 \end{cases}$$

If  $f(x)$  is continuous on  $\mathbb{R}$ , then  $a + b$  equals:

(1)  $-3$       (2)  $-1$       (3)  $3$       (4)  $1$

**Official Ans. by NTA (2)**

**Sol.**  $f(x)$  is continuous on  $\mathbb{R}$

$\Rightarrow f(1^-) = f(1) = f(1^+)$

$2a + 1 + b = \lim_{x \rightarrow 1} \sin(\pi x)$

$2a + 1 + b = 0 \Rightarrow a + b = -1$  ... (1)

$\Rightarrow$  Also  $f(-1^-) = f(-1) = f(-1^+)$

$\lim_{x \rightarrow -1} 2 \sin\left(-\frac{\pi x}{2}\right) = |a - 1 + b|$

$2a - 1 + b = 2$

Either  $a - 1 + b = 2$  or  $a - 1 + b = -2$

$a + b = 3$  ... (2) or  $a + b = -1$  ... (3)

from (1) and (2)  $\Rightarrow a + b = 3 = -1$  (reject)

from (1) and (3)  $\Rightarrow a + b = -1$

4. For  $x > 0$ , if  $f(x) = \int_1^x \frac{\log_e t}{(1+t)} dt$ , then  $f(e) + f\left(\frac{1}{e}\right)$

is equal to

(1)  $1$       (2)  $-1$       (3)  $\frac{1}{2}$       (4)  $0$

**Official Ans. by NTA (3)**

**Sol.**  $f(x) = \int_1^x \frac{\log_e t}{(1+t)} dt$

$f\left(\frac{1}{x}\right) = \int_1^{1/x} \frac{\log_e t}{1+t} dt$ , let  $t = \frac{1}{y}$

$$= + \int_1^x \frac{\ln y}{1+y} \cdot \frac{y}{y^2} dy$$

$$= \int_1^x \frac{\ln y}{y(1+y)} dy$$

hence

$$f(x) + f\left(\frac{1}{x}\right) = \int_1^x \frac{(1+t)\ln t}{t(1+t)} dt = \int_1^x \frac{\ln t}{t} dt$$

$$= \frac{1}{2} \ln^2(x)$$

$$\text{so } f(e) + f\left(\frac{1}{e}\right) = \frac{1}{2} \quad \dots(3)$$

5. A natural number has prime factorization given by  $n = 2^x 3^y 5^z$ , where  $y$  and  $z$  are such that

$$y + z = 5 \text{ and } y^{-1} + z^{-1} = \frac{5}{6}, y > z. \text{ Then the}$$

number of odd divisors of  $n$ , including 1, is :

- (1) 11      (2) 6      (3) 6x      (4) 12

**Official Ans. by NTA (4)**

**Sol. Ans. (4)**

**Sol.**  $y + z = 5$

$$\frac{1}{y} + \frac{1}{z} = \frac{5}{6} \qquad y > z$$

$$\Rightarrow y = 3, z = 2$$

$$\Rightarrow n = 2^x \cdot 3^3 \cdot 5^2 = (2.2.2 \dots) (3.3.3) (5.5)$$

$$\text{Number of odd divisors} = 4 \times 3 = 12$$

6. Let  $f(x) = \sin^{-1}x$  and  $g(x) = \frac{x^2 - x - 2}{2x^2 - x - 6}$ . If

$g(2) = \lim_{x \rightarrow 2} g(x)$ , then the domain of the function  $f \circ g$  is :

(1)  $(-\infty, -2] \cup \left[-\frac{3}{2}, \infty\right)$

(2)  $(-\infty, -2] \cup [-1, \infty)$

(3)  $(-\infty, -2] \cup \left[-\frac{4}{3}, \infty\right)$

(4)  $(-\infty, -1] \cup [2, \infty)$

**Official Ans. by NTA (3)**

**Sol.** Domain of  $f \circ g(x) = \sin^{-1}(g(x))$

$$\Rightarrow |g(x)| \leq 1, \quad g(2) = \frac{3}{7}$$

$$\left| \frac{x^2 - x - 2}{2x^2 - x - 6} \right| \leq 1$$

$$\left| \frac{(x+1)(x-2)}{(2x+3)(x-2)} \right| \leq 1$$

$$\frac{x+1}{2x+3} \leq 1 \text{ and } \frac{x+1}{2x+3} \geq -1$$

$$\frac{x+1-2x-3}{2x+3} \leq 0 \text{ and } \frac{x+1+2x+3}{2x+3} \geq 0$$

$$\frac{x+2}{2x+3} \geq 0 \text{ and } \frac{3x+4}{2x+3} \geq 0$$

$$x \in (-\infty, -2] \cup \left[-\frac{4}{3}, \infty\right)$$

7. The triangle of maximum area that can be inscribed in a given circle of radius 'r' is :

(1) An isosceles triangle with base equal to  $2r$ .

(2) An equilateral triangle of height  $\frac{2r}{3}$ .

(3) An equilateral triangle having each of its side of length  $\sqrt{3}r$ .

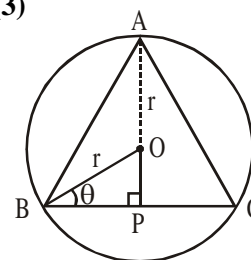
(4) A right angle triangle having two of its sides of length  $2r$  and  $r$ .

**Official Ans. by NTA (3)**

**Sol.**  $h = r \sin \theta + r$

$$\text{base} = BC = 2r \cos \theta$$

$$\theta \in \left[0, \frac{\pi}{2}\right)$$



$$\text{Area of } \Delta ABC = \frac{1}{2} (BC) \cdot h$$

$$\Delta = \frac{1}{2} (2r \cos \theta) \cdot (r \sin \theta + r)$$

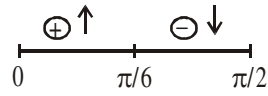
$$= r^2 (\cos \theta) \cdot (1 + \sin \theta)$$

$$\frac{d\Delta}{d\theta} = r^2 [\cos^2 \theta - \sin \theta - \sin^2 \theta]$$

$$= r^2 [1 - \sin \theta - 2\sin^2 \theta]$$

$$= \underbrace{r^2 [1 + \sin \theta]}_{\text{positive}} [1 - 2 \sin \theta] = 0$$

$$\Rightarrow \theta = \frac{\pi}{6}$$



$$\Rightarrow \Delta \text{ is maximum where } \theta = \frac{\pi}{6}$$

$$\Delta_{\text{max.}} = \frac{3\sqrt{3}}{4} r^2 = \text{area of equilateral } \Delta \text{ with}$$

$$BC = \sqrt{3}r.$$

8. Let L be a line obtained from the intersection of two planes  $x + 2y + z = 6$  and  $y + 2z = 4$ . If point  $P(\alpha, \beta, \gamma)$  is the foot of perpendicular from  $(3, 2, 1)$  on L, then the value of  $21(\alpha + \beta + \gamma)$  equals :

- (1) 142      (2) 68      (3) 136      (4) 102

Official Ans. by NTA (4)

Sol.  $x + 2y + z = 6$   
 $(y + 2z = 4) \times 2$

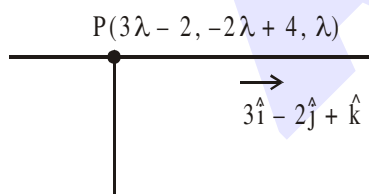
$$x - 3z = -2 \Rightarrow x = 3z - 2 \Rightarrow y = 4 - 2z$$

$$\frac{x+2}{3} = z \qquad \frac{y-4}{-2} = z$$

$\Rightarrow$  line of intersection of two planes is

$$\frac{x+2}{3} = \frac{y-4}{-2} = z = \lambda \quad (\text{Let})$$

$\therefore AP \perp^{\text{ar}}$  to line



$$\therefore \overline{AP} \cdot (3\hat{i} - 2\hat{j} + \hat{k}) = 0$$

$$(3\lambda - 5) \cdot 3 + (-2\lambda + 2) \cdot (-2) + (\lambda - 1) \cdot 1 = 0$$

$$9\lambda - 15 + 4\lambda - 4 + \lambda - 1 = 0$$

$$14\lambda = 20$$

$$\lambda = \frac{10}{7} \Rightarrow P\left(\frac{16}{7}, \frac{8}{7}, \frac{10}{7}\right)$$

$$\Rightarrow \alpha + \beta + \gamma = \frac{16+8+10}{7} = \frac{34}{7}$$

$$21(\alpha + \beta + \gamma) = 102$$

9. Let  $F_1(A, B, C) = (A \wedge \sim B) \vee [\sim C \wedge (A \vee B)] \vee \sim A$  and  $F_2(A, B) = (A \vee B) \vee (B \rightarrow \sim A)$  be two logical expressions. Then :

- (1)  $F_1$  and  $F_2$  both are tautologies  
(2)  $F_1$  is a tautology but  $F_2$  is not a tautology  
(3)  $F_1$  is not tautology but  $F_2$  is a tautology  
(4) Both  $F_1$  and  $F_2$  are not tautologies

Official Ans. by NTA (3)

Sol.  $F_1 : (A \wedge \sim B) \vee [\sim C \wedge (A \vee B)] \vee \sim A$

$$F_2 : (A \vee B) \vee (B \rightarrow \sim A)$$

$$\begin{aligned} F_1 &: \{(A \wedge \sim B) \vee \sim A\} \vee \{(A \vee B) \wedge \sim C\} \\ &: \{(A \vee \sim A) \wedge (\sim A \vee \sim B)\} \vee \{(A \vee B) \wedge \sim C\} \\ &: \{t \wedge (\sim A \vee \sim B)\} \vee \{(A \vee B) \wedge \sim C\} \\ &: (\sim A \vee \sim B) \vee \{(A \vee B) \wedge \sim C\} \\ &: \underbrace{[(\sim A \vee \sim B) \vee (A \vee B)]}_{t} \wedge [(\sim A \vee \sim B) \wedge \sim C] \end{aligned}$$

$$F_1 : (\sim A \vee \sim B) \wedge \sim C \neq t \text{ (tautology)}$$

$$F_2 : (A \vee B) \vee (\sim B \vee \sim A) = t \text{ (tautology)}$$

10. Let slope of the tangent line to a curve at any point  $P(x, y)$  be given by  $\frac{xy^2 + y}{x}$ . If the curve

intersects the line  $x + 2y = 4$  at  $x = -2$ , then the value of  $y$ , for which the point  $(3, y)$  lies on the curve, is :

- (1)  $\frac{18}{35}$       (2)  $-\frac{4}{3}$       (3)  $-\frac{18}{19}$       (4)  $-\frac{18}{11}$

Official Ans. by NTA (3)

Sol.  $\frac{dy}{dx} = \frac{xy^2 + y}{x}$

$$\frac{xdy - ydx}{y^2} = xdx$$

$$-d\left(\frac{x}{y}\right) = xdx$$

$$-\frac{x}{y} = \frac{x^2}{2} + c$$

$\therefore$  curve intersects the line  $x + 2y = 4$  at  $x = -2 \Rightarrow$  point of intersection is  $(-2, 3)$   
 $\therefore$  curve passes through  $(-2, 3)$   
 $\Rightarrow \frac{2}{3} = 2 + c \Rightarrow c = -\frac{4}{3}$

$$\Rightarrow \frac{-x}{y} = \frac{x^2}{2} - \frac{4}{3}$$

Now put  $(3, y)$

$$\Rightarrow \frac{-3}{y} = \frac{19}{6}$$

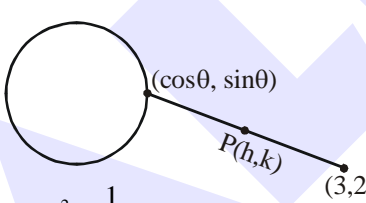
$$\Rightarrow y = \frac{-18}{19}$$

11. If the locus of the mid-point of the line segment from the point  $(3, 2)$  to a point on the circle,  $x^2 + y^2 = 1$  is a circle of radius  $r$ , then  $r$  is equal to :

- (1) 1      (2)  $\frac{1}{2}$       (3)  $\frac{1}{3}$       (4)  $\frac{1}{4}$

Official Ans. by NTA (2)

Sol.  $h = \frac{\cos\theta + 3}{2}$   
 $k = \frac{\sin\theta + 2}{2}$



$\Rightarrow \left(h - \frac{3}{2}\right)^2 + (k - 1)^2 = \frac{1}{4}$   
 $\Rightarrow r = \frac{1}{2}$

12. Consider the following system of equations :

$$x + 2y - 3z = a$$

$$2x + 6y - 11z = b$$

$$x - 2y + 7z = c,$$

where  $a, b$  and  $c$  are real constants. Then the system of equations :

- (1) has a unique solution when  $5a = 2b + c$   
 (2) has infinite number of solutions when  $5a = 2b + c$   
 (3) has no solution for all  $a, b$  and  $c$   
 (4) has a unique solution for all  $a, b$  and  $c$

Official Ans. by NTA (2)

Sol.  $P_1 : x + 2y - 3z = a$

$$P_2 : 2x + 6y - 11z = b$$

$$P_3 : x - 2y + 7z = c$$

Clearly

$$5P_1 = 2P_2 + P_3 \quad \text{if } 5a = 2b + c$$

$\Rightarrow$  All the planes sharing a line of intersection  
 $\Rightarrow$  infinite solutions

13. If  $0 < a, b < 1$ , and  $\tan^{-1}a + \tan^{-1}b = \frac{\pi}{4}$ , then the value of

$$(a + b) - \left(\frac{a^2 + b^2}{2}\right) + \left(\frac{a^3 + b^3}{3}\right) - \left(\frac{a^4 + b^4}{4}\right) + \dots$$

is :

- (1)  $\log_e 2$       (2)  $e^2 - 1$   
 (3)  $e$       (4)  $\log_e \left(\frac{e}{2}\right)$

Official Ans. by NTA (1)

Sol.  $\tan^{-1}a + \tan^{-1}b = \frac{\pi}{4} \quad 0 < a, b < 1$

$$\Rightarrow \frac{a + b}{1 - ab} = 1$$

$$a + b = 1 - ab$$

$$(a + 1)(b + 1) = 2$$

$$\text{Now } \left[ a - \frac{a^2}{2} + \frac{a^3}{3} - \dots \right] + \left[ b - \frac{b^2}{2} + \frac{b^3}{3} - \dots \right]$$

$$= \log_e(1 + a) + \log_e(1 + b)$$

( $\because$  expansion of  $\log_e(1 + x)$ )

$$= \log_e[(1 + a)(1 + b)]$$

$$= \log_e 2$$

14. The sum of the series  $\sum_{n=1}^{\infty} \frac{n^2 + 6n + 10}{(2n + 1)!}$  is equal to :

(1)  $\frac{41}{8}e + \frac{19}{8}e^{-1} - 10$

(2)  $\frac{41}{8}e - \frac{19}{8}e^{-1} - 10$

(3)  $\frac{41}{8}e + \frac{19}{8}e^{-1} + 10$

(4)  $-\frac{41}{8}e + \frac{19}{8}e^{-1} - 10$

Official Ans. by NTA (2)

Sol.  $T_n = \frac{n^2 + 6n + 10}{(2n + 1)!} = \frac{4n^2 + 24n + 40}{4 \cdot (2n + 1)!}$

$$= \frac{(2n+1)^2 + 20n + 39}{4 \cdot (2n+1)!}$$

$$= \frac{(2n+1)^2 + (2n+1) \cdot 10 + 29}{4(2n+1)!}$$

$$= \frac{1}{4} \left[ \frac{(2n+1)^2}{(2n+1)(2n)!} + \frac{(2n+1)10}{(2n+1)(2n)!} + \frac{29}{(2n+1)!} \right]$$

$$= \frac{1}{4} \left[ \frac{2n+1}{(2n)!} + \frac{10}{(2n)!} + \frac{29}{(2n+1)!} \right]$$

$$= \frac{1}{4} \left[ \frac{1}{(2n-1)!} + \frac{11}{(2n)!} + \frac{29}{(2n+1)!} \right]$$

$$S_1 = \frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \dots = \frac{e - \frac{1}{e}}{2}$$

$$S_2 = 11 \left[ \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \dots \right] = 11 \left[ \frac{e + \frac{1}{e} - 2}{2} \right]$$

$$S_3 = 29 \left[ \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \dots \right] = 29 \left[ \frac{e - \frac{1}{e} - 2}{2} \right]$$

Now,  $S = \frac{1}{4} [S_1 + S_2 + S_3]$

$$= \frac{1}{4} \left[ \frac{e}{2} - \frac{1}{2e} + \frac{11e}{2} + \frac{11}{2e} + \frac{29e}{2} - \frac{29}{2e} - 4 \right]$$

$$= \frac{41e}{8} - \frac{19}{8e} - 10$$

15. Let  $f(x)$  be a differentiable function at  $x = a$  with

$f'(a) = 2$  and  $f(a) = 4$ . Then  $\lim_{x \rightarrow a} \frac{xf(a) - af(x)}{x - a}$

equals :

- (1)  $2a + 4$                       (2)  $4 - 2a$   
 (3)  $2a - 4$                       (4)  $a + 4$

Official Ans. by NTA (2)

Sol.  $f'(a) = 2, f(a) = 4$

$$\lim_{x \rightarrow a} \frac{xf(a) - af(x)}{x - a}$$

$$\Rightarrow \lim_{x \rightarrow a} \frac{f(a) - af'(x)}{1} \quad (\text{Lopitals rule})$$

$$= f(a) - af'(a) = 4 - 2a$$

16. Let  $A(1, 4)$  and  $B(1, -5)$  be two points. Let  $P$  be a point on the circle  $(x - 1)^2 + (y - 1)^2 = 1$  such that  $(PA)^2 + (PB)^2$  have maximum value, then the points  $P, A$  and  $B$  lie on :

- (1) a straight line              (2) a hyperbola  
 (3) an ellipse                      (4) a parabola

Official Ans. by NTA (1)

Sol.  $P$  be a point on  $(x - 1)^2 + (y - 1)^2 = 1$  so  $P(1 + \cos\theta, 1 + \sin\theta)$

$A(1,4) \quad B(1,-5)$

$(PA)^2 + (PB)^2$

$$= (\cos\theta)^2 + (\sin\theta - 3)^2 + (\cos\theta)^2 + (\sin\theta + 6)^2 = 47 + 6\sin\theta$$

is maximum if  $\sin\theta = 1$

$$\Rightarrow \sin\theta = 1, \cos\theta = 0$$

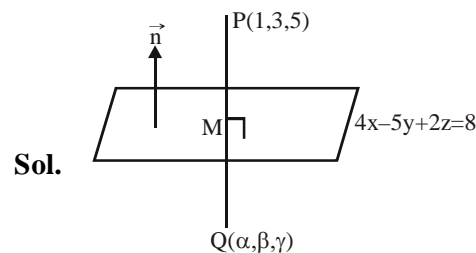
$P(1,1) \quad A(1,4) \quad B(1,-5)$

$P, A, B$  are collinear points.

17. If the mirror image of the point  $(1, 3, 5)$  with respect to the plane  $4x - 5y + 2z = 8$  is  $(\alpha, \beta, \gamma)$ , then  $5(\alpha + \beta + \gamma)$  equals :

- (1) 47              (2) 43              (3) 39              (4) 41

Official Ans. by NTA (1)



Sol.

Point  $Q$  is image of point  $P$  w.r.to plane,  $M$  is mid point of  $P$  and  $Q$ , lies in plane

$$M\left(\frac{1+\alpha}{2}, \frac{3+\beta}{2}, \frac{5+\gamma}{2}\right)$$

$$4x - 5y + 2z = 8$$

$$4\left(\frac{1+\alpha}{2}\right) - 5\left(\frac{3+\beta}{2}\right) + 2\left(\frac{5+\gamma}{2}\right) = 8 \quad \dots(1)$$

Also PQ perpendicular to the plane

$$\Rightarrow \overrightarrow{PQ} \parallel \vec{n}$$

$$\frac{\alpha-1}{4} = \frac{\beta-3}{-5} = \frac{\gamma-5}{2} = k \quad (\text{let})$$

$$\left. \begin{aligned} \alpha &= 1 + 4k \\ \beta &= 3 - 5k \\ \gamma &= 5 + 2k \end{aligned} \right\} \dots(2)$$

use (2) in (1)

$$2(1+4k) - 5\left(\frac{6-5k}{2}\right) + (10+2k) = 8$$

$$k = \frac{2}{5}$$

$$\text{from (2) } \alpha = \frac{13}{5}, \beta = 1, \gamma = \frac{29}{5}$$

$$5(\alpha + \beta + \gamma) = 13 + 5 + 29 = 47$$

18. Let  $f(x) = \int_0^x e^t f(t) dt + e^x$  be a differentiable

function for all  $x \in \mathbb{R}$ . Then  $f(x)$  equals :

(1)  $2e^{(e^x-1)} - 1$                       (2)  $e^{e^x} - 1$

(3)  $2e^{e^x} - 1$                       (4)  $e^{(e^x-1)}$

**Official Ans. by NTA (1)**

**Sol.**  $f(x) = \int_0^x e^t f(t) dt + e^x \Rightarrow f(0) = 1$

differentiating with respect to  $x$

$$f'(x) = e^x f(x) + e^x$$

$$f'(x) = e^x(f(x) + 1)$$

$$\int_0^x \frac{f'(x)}{f(x)+1} dx = \int_0^x e^x dx$$

$$\ln(f(x)+1) \Big|_0^x = e^x \Big|_0^x$$

$$\ln(f(x)+1) - \ln(f(0)+1) = e^x - 1$$

$$\ln\left(\frac{f(x)+1}{2}\right) = e^x - 1 \quad \{\text{as } f(0) = 1\}$$

$$f(x) = 2e^{(e^x-1)} - 1$$

19. Let  $A_1$  be the area of the region bounded by the curves  $y = \sin x$ ,  $y = \cos x$  and  $y$ -axis in the first quadrant. Also, let  $A_2$  be the area of the region bounded by the curves  $y = \sin x$ ,

$y = \cos x$ ,  $x$ -axis and  $x = \frac{\pi}{2}$  in the first quadrant.

Then,

(1)  $A_1 : A_2 = 1 : \sqrt{2}$  and  $A_1 + A_2 = 1$

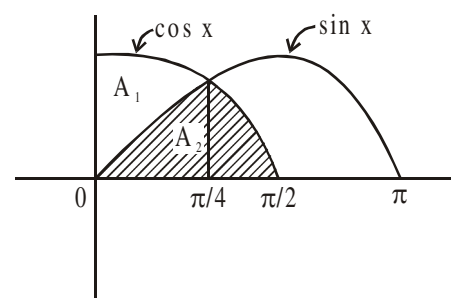
(2)  $A_1 = A_2$  and  $A_1 + A_2 = \sqrt{2}$

(3)  $2A_1 = A_2$  and  $A_1 + A_2 = 1 + \sqrt{2}$

(4)  $A_1 : A_2 = 1 : 2$  and  $A_1 + A_2 = 1$

**Official Ans. by NTA (1)**

**Sol.**



$$A_1 = \int_0^{\pi/4} (\cos x - \sin x) dx$$

$$A_1 = (\sin x + \cos x) \Big|_0^{\pi/4} = \sqrt{2} - 1$$

$$A_2 = \int_0^{\pi/4} \sin x dx + \int_{\pi/4}^{\pi/2} \cos x dx$$

$$= (-\cos x)_{\pi/4}^{\pi/4} + (\sin x)_{\pi/4}^{\pi/2}$$

$$A_2 = \sqrt{2}(\sqrt{2}-1)$$

$$A_1 : A_2 = 1 : \sqrt{2}, A_1 + A_2 = 1$$

20. A seven digit number is formed using digits 3, 3, 4, 4, 4, 5, 5. The probability, that number so formed is divisible by 2, is :

- (1)  $\frac{6}{7}$       (2)  $\frac{1}{7}$       (3)  $\frac{3}{7}$       (4)  $\frac{4}{7}$

**Official Ans. by NTA (3)**

**Sol.** Digits = 3, 3, 4, 4, 5, 5

$$\text{Total 7 digit numbers} = \frac{7!}{2!2!3!}$$

Number of 7 digit number divisible by 2  
 $\Rightarrow$  last digit = 4



3, 3, 4, 4, 5, 5

Now 7 digit numbers which are divisible by 2

$$= \frac{6!}{2!2!2!}$$

$$\text{Required probability} = \frac{\frac{6!}{2!2!2!}}{\frac{7!}{2!2!3!}} = \frac{3}{7}$$

### SECTION B

1. Let  $z$  be those complex numbers which satisfy  $|z + 5| \leq 4$  and  $z(1+i) + \bar{z}(1-i) \geq -10, i = \sqrt{-1}$ .  
 If the maximum value of  $|z + 1|^2$  is  $\alpha + \beta\sqrt{2}$ , then the value of  $(\alpha + \beta)$  is \_\_\_\_\_.

**Official Ans. by NTA (48)**

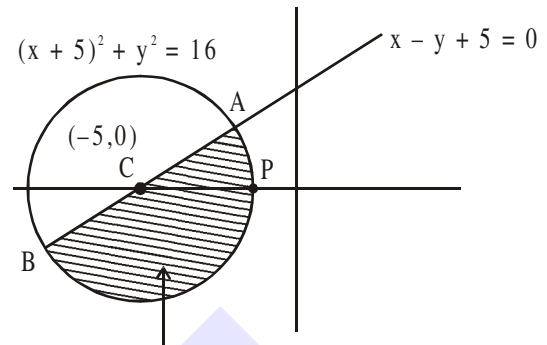
**Sol.**  $|z + 5| \leq 4$

$$(x + 5)^2 + y^2 \leq 16 \quad \dots (1)$$

$$z(1+i) + \bar{z}(1-i) \geq -10$$

$$(z + \bar{z}) + i(z - \bar{z}) \geq -10$$

$$x - y + 5 \geq 0 \quad \dots (2)$$



Region bounded by inequalities (1) & (2)

$$|z + 1|^2 = |z - (-1)|^2$$

Let  $P(-1, 0)$

$|z + 1|_{\text{Max.}}^2 = PB^2$  (where B is in 3<sup>rd</sup> quadrant)  
 for point of intersection

$$\left. \begin{aligned} (x + 5)^2 + y^2 &= 16 \\ x - y + 5 &= 0 \end{aligned} \right\} y = \pm 2\sqrt{2}$$

$$A(2\sqrt{2} - 5, 2\sqrt{2}) \quad B(-2\sqrt{2} - 5, -2\sqrt{2})$$

$$PB^2 = (+2\sqrt{2} + 4)^2 + (2\sqrt{2})^2$$

$$|z + 1|^2 = 8 + 16 + 16\sqrt{2} + 8$$

$$\alpha + \beta\sqrt{2} = 32 + 16\sqrt{2}$$

$$\alpha = 32, \beta = 16 \Rightarrow \alpha + \beta = 48$$

2. Let the normals at all the points on a given curve pass through a fixed point  $(a, b)$ . If the curve passes through  $(3, -3)$  and  $(4, -2\sqrt{2})$ , and given that  $a - 2\sqrt{2}b = 3$ , then  $(a^2 + b^2 + ab)$  is equal to \_\_\_\_\_.

**Official Ans. by NTA (9)**

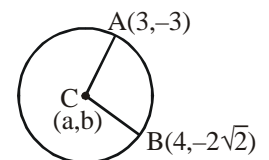
**Sol.** All normals of circle passes through centre

$$\text{Radius} = CA = CB$$

$$CA^2 = CB^2$$

$$(a - 3)^2 + (b + 3)^2$$

$$= (a - 4)^2 + (b - 2\sqrt{2})^2$$



$$a + (3 - 2\sqrt{2})b = 3$$

$$a - 2\sqrt{2}b + 3b = 3 \quad \dots(1)$$

$$\text{given that } a - 2\sqrt{2}b = 3 \quad \dots(2)$$

$$\text{from (1) \& (2) } \Rightarrow a = 3, b = 0$$

$$a^2 + b^2 + ab = 9$$

3. Let  $\alpha$  and  $\beta$  be two real numbers such that  $\alpha + \beta = 1$  and  $\alpha\beta = -1$ . Let  $p_n = (\alpha)^n + (\beta)^n$ ,  $p_{n-1} = 11$  and  $p_{n+1} = 29$  for some integer  $n \geq 1$ . Then, the value of  $p_n^2$  is \_\_\_\_\_.

**Official Ans. by NTA (324)**

**Sol.**  $x^2 - x - 1 = 0$  roots =  $\alpha, \beta$

$$\alpha^2 - \alpha - 1 = 0 \Rightarrow \alpha^{n+1} = \alpha^n + \alpha^{n-1}$$

$$\beta^2 - \beta - 1 = 0 \Rightarrow \beta^{n+1} = \beta^n + \beta^{n-1}$$

$$\begin{array}{r}
 + \\
 \hline
 P_{n+1} = P_n + P_{n-1} \\
 29 = P_n + 11 \\
 P_n = 18 \\
 P_n^2 = 324
 \end{array}$$

4. If  $I_{m,n} = \int_0^1 x^{m-1} (1-x)^{n-1} dx$ , for  $m, n \geq 1$  and

$$\int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx = \alpha I_{m,n}, \alpha \in \mathbb{R}, \text{ then } \alpha \text{ equals}$$

\_\_\_\_\_.

**Official Ans. by NTA (1)**

**Sol.**  $I_{m,n} = \int_0^1 x^{m-1} (1-x)^{n-1} dx = I_{n,m}$

$$\text{Now Let } x = \frac{1}{y+1} \Rightarrow dx = -\frac{1}{(y+1)^2} dy$$

so

$$I_{m,n} = -\int_{\infty}^0 \frac{1}{(y+1)^{m-1}} \frac{y^{n-1}}{(y+1)^{n-1}} \frac{dy}{(y+1)^2} = \int_0^{\infty} \frac{y^{n-1}}{(1+y)^{m+n}} dy$$

$$\text{similarly } I_{m,n} = \int_0^{\infty} \frac{y^{m-1}}{(1+y)^{m+n}} dy$$

$$\text{Now } 2I_{m,n} = \int_0^{\infty} \frac{y^{m-1} + y^{n-1}}{(1+y)^{m+n}} dy$$

$$= \int_0^{\infty} \frac{y^{m-1} + y^{n-1}}{(1+y)^{m+n}} dy$$

$$= \int_0^1 \frac{y^{m-1} + y^{n-1}}{(1+y)^{m+n}} dy + \underbrace{\int_1^{\infty} \frac{y^{m-1} + y^{n-1}}{(1+y)^{m+n}} dy}_{\text{substitute } y = \frac{1}{t}}$$

$$\Rightarrow 2I_{m,n} = \int_0^1 \frac{y^{m-1} + y^{n-1}}{(1+y)^{m+n}} dy - \int_1^0 \frac{t^{n-1} + t^{m-1}}{t^{m+n-2}} \frac{t^{m+n}}{(1+t)^{m+n}} \frac{dt}{t^2}$$

$$\Rightarrow \text{Hence } 2I_{m,n} = 2 \int_0^1 \frac{y^{m-1} + y^{n-1}}{(1+y)^{m+n}} dy \Rightarrow \alpha = 1$$

5. If the arithmetic mean and geometric mean of the  $p^{\text{th}}$  and  $q^{\text{th}}$  terms of the sequence  $-16, 8, -4, 2, \dots$  satisfy the equation  $4x^2 - 9x + 5 = 0$ , then  $p + q$  is equal to \_\_\_\_\_.

**Official Ans. by NTA (10)**

**Sol.**  $4x^2 - 9x + 5 = 0 \Rightarrow x = 1, \frac{5}{4}$

$$\text{Now given } \frac{5}{4} = \frac{t_p + t_q}{2}, t = t_p t_q \text{ where}$$

$$t_r = -16 \left( -\frac{1}{2} \right)^{r-1}$$

$$\text{so } \frac{5}{4} = -8 \left[ \left( -\frac{1}{2} \right)^{p-1} + \left( -\frac{1}{2} \right)^{q-1} \right]$$

$$1 = 256 \left( -\frac{1}{2} \right)^{p+q-2} \Rightarrow 2^{p+q-2} = (-1)^{p+q-2} 2^8$$

$$\text{hence } p + q = 10$$



6. The total number of 4-digit numbers whose greatest common divisor with 18 is 3, is \_\_\_\_\_.

**Official Ans. by NTA (1000)**

- Sol.** Let N be the four digit number  
 $\gcd(N, 18) = 3$   
 Hence N is an odd integer which is divisible by 3 but not by 9.

4 digit odd multiples of 3

1005, 1011,....., 9999  $\rightarrow$  1500

4 digit odd multiples of 9

1017, 1035,....., 9999  $\rightarrow$  500

Hence number of such N = 1000

7. Let L be a common tangent line to the curves  $4x^2 + 9y^2 = 36$  and  $(2x)^2 + (2y)^2 = 31$ . Then the square of the slope of the line L is \_\_\_\_\_.

**Official Ans. by NTA (3)**

- Sol.** Given curves are  $\frac{x^2}{9} + \frac{y^2}{4} = 1$

$$x^2 + y^2 = \frac{31}{4}$$

let slope of common tangent be m

so tangents are  $y = mx \pm \sqrt{9m^2 + 4}$

$$y = mx \pm \frac{\sqrt{31}}{2} \sqrt{1 + m^2}$$

$$\text{hence } 9m^2 + 4 = \frac{31}{4}(1 + m^2)$$

$$\Rightarrow 36m^2 + 16 = 31 + 31m^2 \Rightarrow m^2 = 3$$

8. Let a be an integer such that all the real roots of the polynomial  $2x^5 + 5x^4 + 10x^3 + 10x^2 + 10x + 10$  lie in the interval  $(a, a + 1)$ . Then,  $|a|$  is equal to \_\_\_\_\_.

**Official Ans. by NTA (2)**

- Sol.** Let  $2x^5 + 5x^4 + 10x^3 + 10x^2 + 10x + 10 = f(x)$

Now  $f(-2) = -34$  and  $f(-1) = 3$

Hence  $f(x)$  has a root in  $(-2, -1)$

Further  $f'(x) = 10x^4 + 20x^3 + 20x^2 + 20x + 10$

$$= 10x^2 \left[ \left( x^2 + \frac{1}{x^2} \right) + 2 \left( x + \frac{1}{x} \right) + 20 \right]$$

$$= 10x^2 \left[ \left( x + \frac{1}{x} + 1 \right)^2 + 17 \right] > 0$$

Hence  $f(x)$  has only one real root, so  $|a| = 2$

9. Let  $X_1, X_2, \dots, X_{18}$  be eighteen observations

such that  $\sum_{i=1}^{18} (X_i - \alpha) = 36$  and

$$\sum_{i=1}^{18} (X_i - \beta)^2 = 90, \text{ where } \alpha \text{ and } \beta \text{ are distinct}$$

real numbers. If the standard deviation of these observations is 1, then the value of  $|\alpha - \beta|$  is \_\_\_\_\_.

**Official Ans. by NTA (4)**

- Sol.**  $\sum_{i=1}^{18} (x_i - \alpha) = 36, \sum_{i=1}^{18} (x_i - \beta)^2 = 90$

$$\Rightarrow \sum_{i=1}^{18} x_i = 18(\alpha + 2), \sum_{i=1}^{18} x_i^2 - 2\beta \sum_{i=1}^{18} x_i + 18\beta^2 = 90$$

$$\text{Hence } \sum_{i=1}^{18} x_i^2 = 90 - 18\beta^2 + 36\beta(\alpha + 2)$$

$$\text{Given } \frac{\sum_{i=1}^{18} x_i^2}{18} - \left( \frac{\sum_{i=1}^{18} x_i}{18} \right)^2 = 1$$

$$\Rightarrow 90 - 18\beta^2 + 36\beta(\alpha + 2) - 18(\alpha + 2)^2 = 18$$

$$\Rightarrow 5 - \beta^2 + 2\alpha\beta + 4\beta - \alpha^2 - 4\alpha - 4 = 1$$

$$\Rightarrow (\alpha - \beta)^2 + 4(\alpha - \beta) = 0 \Rightarrow |\alpha - \beta| = 0 \text{ or } 4$$

As  $\alpha$  and  $\beta$  are distinct  $|\alpha - \beta| = 4$

10. If the matrix  $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 0 & -1 \end{bmatrix}$  satisfies the

equation  $A^{20} + \alpha A^{19} + \beta A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$  for

some real numbers  $\alpha$  and  $\beta$ , then  $\beta - \alpha$  is equal to \_\_\_\_\_.

**Official Ans. by NTA (4)**

**Sol.**  $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 0 & -1 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 3 & 0 & -1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence

$$A^{20} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^{20} & 0 \\ 0 & 0 & 1 \end{bmatrix}, A^{19} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^{19} & 0 \\ 3 & 0 & -1 \end{bmatrix}$$

$$\text{So } A^{20} + \alpha A^{19} + \beta A = \begin{bmatrix} 1+\alpha+\beta & 0 & 0 \\ 0 & 2^{20} + \alpha \cdot 2^{19} + 2\beta & 0 \\ 3\alpha+3\beta & 0 & 1-\alpha-\beta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore  $\alpha + \beta = 0$  and  $2^{20} + 2^{19}\alpha - 2\alpha = 4$

$$\Rightarrow \alpha = \frac{4(1-2^{18})}{2(2^{18}-1)} = -2$$

hence  $\beta = 2$   
so  $(\beta - \alpha) = 4$